

HISTOIRE DU THEOREME DES ACCROISSEMENTS FINIS. By Pierre Dugac. Paris (Université Pierre et Marie Curie). 1979. 60 pp., reproduced typescript;

LIMITE, POINT D'ACCUMULATION, COMPACT. By Pierre Dugac. Paris (Université Pierre et Marie Curie). 1980. 80 pp., reproduced typescript.

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Nineteenth-century mathematics saw the building of a rigorous foundation for analysis. In the two works under review, Pierre Dugac uses books, correspondence, and unpublished mathematical lectures to produce an interesting introduction to some important topics in the rigorization of 19th-century analysis. Neither strictly "histories" nor source books, these works, based on courses given by their author at the Université Pierre et Marie Curie, are a mixture of the two forms: they give long quotations, with French translations, from primary sources, together with a substantial, historically informed connecting narrative. The narrative in these books treats 19th-century rigor as having developed progressively, where pioneering works with gaps were succeeded by later works which eventually filled those gaps: Dugac portrays no revolutions, and few conjectures and refutations.

The first work focuses both on the history of the mean-value theorem (commonly called, in French, the *théorème des accroissements finis* [1]), and on the history of results about the real numbers needed to give a correct proof of the theorem assuming only that the function is differentiable. Dugac reproduces the first and subsequent major treatments not only of the mean-value theorem, but also of Rolle's theorem, the Bolzano-Weierstrass theorem, the monotone-sequence property, the theorem that a bounded set of real numbers has a least upper bound, the theorem that a function continuous on a closed interval attains its maximum, the Cauchy criterion, and the construction of the real numbers as Cauchy sequences of rationals. Dugac first quotes the mean-value theorem in geometric form as stated by Cavalieri (1635); then reproduces Rolle's algebraic statement and proof of Rolle's theorem (1690); gives Lagrange's attempt (1797) to prove that a function with positive derivative on an interval is increasing there, and Lagrange's use of that theorem to prove the mean-value theorem. Dugac points out that Lagrange's proofs implicitly assume the intermediate-value theorem for continuous

functions, that the given function is differentiable, and that its derivative is continuous on the interval; but he observes also that Lagrange's theorems and proofs mark a major step in increasing rigor in analysis.

Next, Dugac takes up Bolzano's 1817 paper proving the intermediate-value theorem for continuous functions; he quotes from that paper both Bolzano's formulation of what is usually called the Cauchy criterion (Cauchy's version having been published in 1821) and Bolzano's proof, assuming the "Cauchy" criterion, that a bounded set of real numbers has a least upper bound. Dugac then returns to the mean-value theorem: he reviews the proofs by Ampère (1806) and Cauchy (1823), showing their dependence on Lagrange's proof; then he reproduces the proof by Ossian Bonnet (1868) which did away with the hypothesis of the continuity of the derivative, but which implicitly assumed that every function was piecewise monotonic; he describes the proof of C. Jordan (1882) and Peano's criticism that Jordan implicitly assumed the uniform convergence of the difference quotient to its limit; finally, he quotes Dini's proofs (which Peano commended to Jordan) of 1878, not only of the mean-value theorem, but of the theorems, attributed by Dini to Weierstrass, that a bounded set of real numbers has a least upper bound, and that a function continuous on a closed interval is bounded. Dugac remarks that Dini's method of proving this last theorem "recalls that of Bolzano," and comparing the texts of the proofs as quoted by Dugac will demonstrate this to the reader. Dini's use of Dedekind cuts for his proofs is also described.

The resemblances between Dini's work and that of Bolzano and Weierstrass raise an important historical question. What, Dugac inquires, were the the roles of Bolzano and Weierstrass in bringing about Dini's demonstrations? Dini himself gives credit to notes on Weierstrass' lectures given him by H. A. Schwarz (presumably those taken by Schwarz in 1861; the manuscript is now at the Institut Mittag-Leffler). In lectures given in 1861, Weierstrass had proved Rolle's theorem under the hypothesis that the function is not only differentiable, but that the derivative is continuous. Apparently Weierstrass never gave a proof of the mean-value theorem assuming only the differentiability of the function, though all the necessary techniques to prove it with this weaker hypothesis--in particular, the theorems that a bounded set of real numbers has a least upper bound, and that a function continuous on a closed interval is bounded and attains its maximum and minimum--were in Weierstrass' lectures at least by 1874.

Dugac traces at length the development of the techniques needed to prove these theorems in Weierstrass' unpublished lectures. He argues that sometime before 1870 Weierstrass had learned from Bolzano's 1817 paper how to prove, using the Cauchy criterion, that a bounded set of real numbers has a least upper

bound. As part of the evidence, Dugac cites a letter of April 1, 1870, from H. A. Schwarz to Georg Cantor saying that both Bolzano and Weierstrass deserve credit for the method of proving that a function continuous on a closed interval attains its maximum, praising Bolzano's 1817 paper, and concluding:

Auch ich bekenne mich mit Dir zu der von Herrn Weierstrass in seinen Vorlesungen verfochtenen Meinung, dass man ohne die Schlussweise, welche von Herrn W. auf Bolzanoschen Principien weiter ausgebildet ist, bei vielen Untersuchungen nicht zum Ziel gelangen konnte. [2]

Also, Dugac reproduces, from a set of notes on Weierstrass' 1874 lectures (taken by G. Hettner; MS now at Göttingen), Weierstrass' proof that a bounded set of real numbers has a least upper bound. Dugac characterizes this as showing "the same inspiration" as Bolzano's--visible to the reader who compares the two proofs. Dugac observes that in these 1874 notes there is a reference to Bolzano's 1817 paper.

Indeed, Dugac argues that Weierstrass knew Bolzano's 1817 paper even earlier, since in 1865-1866 Weierstrass' lecture notes (taken by M. Pasch; MS now at Giessen) include the theorem that a bounded infinite set of points in the same plane has at least one cluster point. This, says Dugac, closely resembles a theorem of Bolzano's. Furthermore, in the second of the books under review, Dugac (p. 52) cites an explicit reference to Bolzano's 1817 paper in Weierstrass' 1865-1866 lecture notes. Thus Bolzano's work was known to Weierstrass earlier than has usually been thought, and Dugac argues that Bolzano-style proof methods had real influence on Weierstrass' work--and thus on the history of the mean-value theorem.

The second of the books under review treats the history of the concept of limit, and--in briefer compass--those of cluster point (*point d'accumulation*, limit point) and compactness. The first section is on limits. Proceeding chronologically, it gives primary-source excerpts which define limit, function, sum of series, convergence, derivative, and integral, from the work of D'Alembert (seen by Dugac as the first to understand the limit concept clearly), through Euler (on the concept of function), L'Huilier (the first to treat variables oscillating about their limits in his definition), Lagrange (who, though claiming to avoid the notion of limit, gave inequality proofs which in fact implicitly use limits), Lacroix (whose calculus texts touch on, though they do not definitively treat, the problem of changing the order of taking limits). Dugac goes on to treat Bolzano's work on continuity and series, Cauchy's definition of limit and his clear sense of what the limit concept means in terms of in-

equalities, and Cauchy's treatments of singular values of functions and convergence of series. The section on limits closes with selections from Dirichlet's work on convergence of trigonometric series and the concept of function, and a statement of the crucial importance of Dirichlet's work both for these problems and for the study of the structure of the set of points of discontinuity of a function.

The second section of this book deals with cluster points. Since Weierstrass introduced the concept of cluster point to treat the principle of analytic continuation, Dugac begins with a modern exposition of results about the zeros of analytic functions. He then traces the history of the idea of cluster point, beginning with the work of Briot and Bouquet (1859; translated into German in 1862) on the zeros of analytic functions, but quickly turning to the work of Weierstrass, concerning which Dugac repeats some of the major conclusions from his book on the mean-value theorem. He then explains how Cantor used Weierstrass' work to study sets, their subsets of measure zero, derived sets, and well-ordering; Cantor's work is summarized, rather than being presented through extensive quotations. There is also some mention of the work of Dedekind. The last section of Dugac's book briefly traces the concept of compactness through the unpublished work of Weierstrass and the publications of Heine, Pincherle, Borel, and Lebesgue, closing with a short description of Fréchet's work on various properties equivalent to compactness. Finally, as an appendix to this book, Dugac gives Bolzano's proof that a bounded set of real numbers has a least upper bound, photocopied from a French translation of Bolzano's 1817 paper [3].

These works, especially the first book and the first section of the second, reproduce and discuss portions of important texts and unpublished materials. By concentration on specific topics, Dugac displays the historical development of proof methods as well as that of specific concepts. However, the consequent narrowness of focus in the narrative means that the reader does not always get a sense of how the specific concepts and results fit into the mathematics of the time, other than with the specific concepts emphasized by Dugac. For instance, little sense is given of why Lagrange or Cauchy might have wanted the mean-value theorem; the place of various notions of completeness of the real numbers in the work of Weierstrass and his school and of Dedekind is not fully elucidated.

The documented influence of Bolzano's 1817 paper on Weierstrass and his school between 1865 and 1870 is an interesting new finding. Perhaps, since Dugac's books are lecture notes, they are not intended to be a systematic exposition of the historiographical impact of their new discoveries. But Dugac's finding challenges the usual picture of the rediscovery of Bolzano's previously uninfluential work by men like Hankel

and Stolz in the 1870s and 1880s. Dugac does not discuss why Bolzano's work was not mentioned in the published literature between 1817 and 1870, nor does he even allude to the question of historical relationship--if any--between Cauchy and Bolzano. The influence of Bolzano on Weierstrass also raises other questions for future research. It would be interesting to pursue the details of the process by which Bolzano's work became known. With whom, in what terms, and when, might Weierstrass have discussed it? And how much did Weierstrass really need what he read in Bolzano's paper?

Anyone studying or working on 19th-century analysis will want to consult Dugac's books. It is of real value to have identified and made accessible the classic expositions, including Weierstrass' unpublished lectures, of these important concepts and theorems. Dugac's discussion of the importance of the proofs of Lagrange, his demonstration that Dini's was the first correct proof of the mean-value theorem, and his treatment of the influence of Bolzano on Weierstrass present welcome new emphases to guide future and fuller interpretations. Dugac's use of Weierstrass' unpublished lecture notes to study the development of the ideas of Weierstrass and his school (begun in [Dugac 1973]) promises important future discoveries. Meanwhile, scholars and students of the history of mathematics will profit both from Dugac's guidance and from the encounter, provided by these books, with original sources.

NOTES

1. On the various names for this theorem in English and French, see R. E. Hedrick's remarks in his translation of Goursat's *Cours d'analyse* [Goursat 1904, Vol. 1, 8 n].
2. "I also agree, like you, with the opinion, upheld by Herr Weierstrass in his lectures, that without the conclusions which Herr W. developed from Bolzano's principles, one could not succeed in many researches" (Reviewer's translation). Quoted by Dugac, p. 47, from [Meschkowski 1967, 228-229].
3. [Bolzano 1964, 153-157]. For an English translation of Bolzano's 1817 paper, see [Russ 1980].

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